Practice #3 - Span and Linear Independence

1. Consider the following set of vectors.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   (a) Verify that the set spans $\mathcal{R}^3$ by checking each of the three criteria needed to determine whether a set of vectors spans $\mathcal{R}^3$.

      i. Criteria 1:

      ii. Criteria 2:

      iii. Criteria 3:

   (b) How many vectors can you remove from the set and still guarantee that the set spans $\mathcal{R}^3$?

   (c) Which vectors would you remove so the remaining vectors still span $\mathcal{R}^3$. **Hint:** Row-reduce the matrix whose columns are the vectors in the set. Which columns can you safely remove and still guarantee that

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + \ldots + x_m\mathbf{u}_m = \mathbf{b} \tag{1}$$

   has a solution for every $\mathbf{b}$?

2. A set of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m$ spans $\mathcal{R}^n$ *if and only if* the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + \ldots + x_m\mathbf{u}_m = \mathbf{b} \tag{2}$$

   has a solution for every vector $\mathbf{b} \in \mathcal{R}^n$.

   - This definition allows for *more than one possible solution.* Why?
   - Formulate a strategy for removing vectors, if necessary, so that the solution is unique.

3. Suppose you are given a set of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots \mathbf{u}_m$. Define a criteria under which the solution to

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + \ldots + x_m\mathbf{u}_m = \mathbf{0} \tag{3}$$

is unique. What is this unique solution?

If this criteria is met, we say that the vectors are *linearly independent.*

4. Is the following set of vectors is linearly independent? Do they span $\mathcal{R}^3$?

$$\left\{ \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -8 \\ 2 \\ -19 \end{bmatrix} \right\}$$

Form the matrix A from the vectors above. How many solutions does the system $A\mathbf{x} = \mathbf{b}$, for any vector $\mathbf{b} \in \mathcal{R}^3$, have?

5. Is the following set of vectors is linearly independent? Do they span $\mathcal{R}^3$?

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} \right\}$$

Form the matrix A from the vectors above. How many solutions does the system $A\mathbf{x} = \mathbf{b}$, for any vector $\mathbf{b} \in \mathcal{R}^3$, have?