Review for Final - Linear Algebra

In addition to this review, also review any Practice worksheets we have done in class.

1. **Concepts. Gaussian Elimination.** Know how to use Gaussian Elimination in matrix form to solve the linear system $A\mathbf{x} = \mathbf{b}$.

   (a) Construct an **augmented matrix** from A and $\mathbf{b}$

   (b) Reduce the augmented matrix to **echelon** form.

   (c) Identify **pivot variables** and **free variables**.

   (d) Solve the resulting system by back substitution.

   (e) Write down the **solution** to the linear system, using free variables if necessary.

   Example. Solve $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination.

$$A = \begin{bmatrix} 1 & -1 & -3 & -1 \\ -2 & 2 & 6 & 2 \\ -3 & -3 & 10 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$$

2. **Gaussian Jordan Elimination.** Use Gauss Jordan Elimination to get the augmented matrix into **reduced row echelon form**.

   (a) Be able to distinguish *reduced echelon form* from just *echelon form.*

   (b) Use a *forward phase* to reduce an augmented matrix to echelon form.

   (c) Use a *backward phase* to reduce to reduced row echelon form.

   (d) Use Gauss Jordan elimination to solve a homogeneous linear system.
   Example. Show that the matrix A above reduces to

$$A \sim B = \begin{bmatrix} 1 & 0 & -19/6 & -1/2 \\ 0 & 1 & -1/6 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   Use this to solve the linear system $A\mathbf{x} = 0$. Write your solution in vector form.

   (e) Use Gauss-Jordan elimination to find the inverse of a square, invertible matrix.
   Example. Verify the following for matrix A :

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

   This matrix is already in upper-triangular form, so you only have to apply the backward phase of Gauss-Jordan elimination to solve for the inverse.
   Example. Find the inverse of A using Gauss-Jordan elimination

$$A = \begin{bmatrix} 2 & -5 & 1 \\ -1 & 3 & -1 \\ 1 & -3 & 4 \end{bmatrix}$$

   (f) Use Gauss Jordan elimination to find the null space of a matrix.
   Example. Use the row-reduced echelon form of A from Problem 2d, above, to find the null space of A.

$$\text{null}(A) = \text{span}\,\{\ldots\}$$

   (g) Use Gauss Jordan elimination to find eigenvectors of a square matrix.
   Example. Find the eigenvectors associated with the eigenvalues $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$ of the matrix below

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 2 & 4 & 2 \end{bmatrix}$$

Example: Use Gauss-Jordan Elimination to solve the linear system from Problem 1. Express the final solution as an *inhomogenous solution* plus a null space solution.

$$A = \begin{bmatrix} 1 & -1 & -3 & -1 \\ -2 & 2 & 6 & 2 \\ -3 & -3 & 10 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$$

3. **Subspaces.** You should know the following concepts.

   (a) Linear combination

   (b) Span

   (c) Euclidean spaces $\mathcal{R}^2$, $\mathcal{R}^3$, and so on.

   Example. Determine if the columns of A span R^3.

$$A = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 1 & 4 & 2 & 6 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

   Example. Answer True/False.

   (a) Any three vectors in $\mathcal{R}^3$ will span R^3.

   (b) Any set of linear independndt vectors in R^n will span R^n.

   (c) The columns of an invertible $n \times n$ matrix will span $\mathcal{R}^n$.

   (d) If $A\mathbf{x} = \mathbf{b}$ has more than one solution for a vector $\mathbf{b}$, then the columns of A must be linearly independent.

   (e) If A is a matrix with more rows than columns, then the columns of A are linearly independent.

4. **Linear transformations.** Know the connection between linear systems $A\mathbf{x} = \mathbf{b}$ and linear transformations $T(\mathbf{x}) = A\mathbf{x}$.

   (a) What is the defining feature of a linear transformation?

   (b) Can all linear transformations be written as $T(\mathbf{x}) = A\mathbf{x}$?

   (c) Use the echelon form of a matrix to identify when a transformation is one-to-one or onto. Can a transformation be both one-to-one and onto? Can it be neither one-to-one nor onto? Provide examples of echelon patterns for each of four possible cases.

   (d) Be able to find a linear transformation given input/output data. (Think of this problem as analogous to finding the equation of a line given two points on the line.)

   Example. Find a transformation $T(\mathbf{x}) = A\mathbf{x}$ so that

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

   Example. True/False

   (a) A linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one and onto if and only if A is invertible.

5. **Subspaces.** Be able to do the following.

   (a) Find bases for the column space, row space and null space of a matrix.

   (b) Identify the dimension of a space spanned by a set of vectors

   (c) Verify the rank-nullity theorem.

6. **Determinants.**

   (a) Compute the determinant of 3×3 and 4×4 matrix.

   (b) Be able to quickly find determinants of larger matrices by exploiting special structures.

Example. Compute the determinant of the following matrices.

(a) Matrix A is upper triangular. How would you compute its determinant?

(b) Matrix B is lower triangular. How would you compute its determinant?

(c) Matrix B can be formed from A by doing elementary row operations (without any rows swaps). How are the determinants of A and B related?

(d) Suppose $A = \text{diag}(1, 2, a, 5, 6)$. For what values of a is A invertible?

(e) Suppose A and B are $n \times n$ matrices and the matrix A is singular. What is $\det(AB)$?

(f) Suppose $\det(A) = 5$. What is $\det(A^{-1})$?

(g) Suppose A is 4×4 and $\det(A) = -3$. What is $\det(2A)$?

(h) Suppose A has an eigenvalue $\lambda = -2$. What is $\det(A + 2I)$?

Example. Find the cofactor matrix of A. Use this to find the inverse of A.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

7. **Eigenvalue/eigenvector problems.**

(a) Be able to find the eigenvalues/eigenvectors of a matrix.

(b) Be able to diagonalize a matrix, if possible.

(c) What is the connection between the multiplicity of an eigenvalue and the dimension of the associated eigenspace?

(d) Compute powers of a diagonalizable matrix.

Example. Diagonalize A if possible.

$$A = \begin{bmatrix} 4 & -1 & -2 \\ -6 & 3 & 4 \\ 8 & -2 & -4 \end{bmatrix}$$

Example. Diagonalize, if possible, the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8. **Unifying Theorems.** Understand all equivalent statements in the Unifying Theorem (Version 8), in section 6.1 of your textbook.