Practice #11 - Linear Algebra

1. Find the basis for the null space of matrix A. Do as little work as possible. *Hint : Swap rows to get the matrix in reduced row echelon form.*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

2. Find the eigenvalues and eigenvectors of the matrix A.

$$\begin{bmatrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 2 & 14 & 4 \end{bmatrix}$$

(a) Write down the *characteristic polynomial* $p(\lambda) = \det(A - \lambda I)$ for A.

(b) Solve $p(\lambda) = 0$ for eigenvalues λ. Hint : You will get three distinct eigenvalues.

(c) For each eigenvalue, solve $\det(A - \lambda I)\mathbf{x} = 0$ for associated eigenvector $\mathbf{x}$.

3. Diagonalize A.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 2 & 4 & 2 \end{bmatrix}$$

(a) Find the eigenvalues and associated eigenvectors of A.

(b) Create a matrix P whose columns are the eigenvectors of A.

(c) Create a diagonal matrix D with diagonal entries equal to the eigenvalues of A. Be sure to align eigenvalues and eigenvectors.

(d) Convince yourself that the following matrix relation holds :

$$AP = PD$$

Hint:

$$A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

(e) The diagonalization can then be written as $A = PDP^{-1}$.

(f) Question : How do we know that P is invertible? Answer : Eigenvectors associated with distinct eigenvalues are linearly independent!

4. Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) What is the multiplicity of each eigenvalue?

(b) What is the eigenspace associated with each eigenvector?

(c) Is A diagonalizable?

5. Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) What is the multiplicity of each eigenvalue?

(b) What is the eigenspace associated with each eigenvector?

(c) Is A diagonalizable?

6. (True/False). A matrix A is diagonalizable if and only if it is invertible. (T/F). If False, explain or find a counter example.

7. Consider the general matrix A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   (a) Write the characteristic polynomial in terms of the *trace* of A given by $\text{tr}(A) = a + d$ and the determinant of A given by $\det(A) = ad - bc$.

   (b) Show that the sum of the eigenvalues of A s equal to $\text{tr}(A)$

   (c) Show that the product of the eigenvalues of A is equal to $\det(A)$.