

Figure 1: Plots of the Airy functions in Problem 1.

Homework #1

Math 427/527

Note : Math 427 students may do the Math 527 questions for extra credit. You may work in pairs on this assignment, but pairs can only be two 427 students or two 527 students but not mixed pairs.

1. The differential equation

$$y'' - xy = 0 \tag{1}$$

is known as the "Airy equation", after the British astronomer and physicist George Biddell Airy (1801-1892). This equation is also the simplest example of an equation with a "turning point" or a point where the character of the solution changes from oscillatory to exponential. This equation is used in modeling particle flows in semiconductor devices. See Wikipedia for more details.

The solutions to the Airy equation are given by

$$y(x) = c_1 \text{Ai}(x) + c_2 \text{Bi}(x) \tag{2}$$

where $\text{Ai}(x)$ is the Airy function and $\text{Bi}(x)$ is the Airy Bi function. In MATLAB, both $\text{Ai}(x)$ and $\text{Bi}(x)$ are defined using `airy`.

- Write out the first 9 terms in the power series solution to the Airy equation. Your final answer should be in terms of a_0 and a_1 . Show your work.
- Plot both the Airy function $\text{Ai}(x)$ and the Airy Bi function $\text{Bi}(x)$ over the interval $[-20, 5]$. See Figure 2 for example of what your plot should look like.

This problem is designed to get you familiar with special functions other than the $\cos(x)$, $\sin(x)$, e^x and so on. Like these more familiar functions, the Airy functions are just solutions to particular ODEs that arise frequently in mathematical models of physical systems. You'll see this function again in Problem 3.

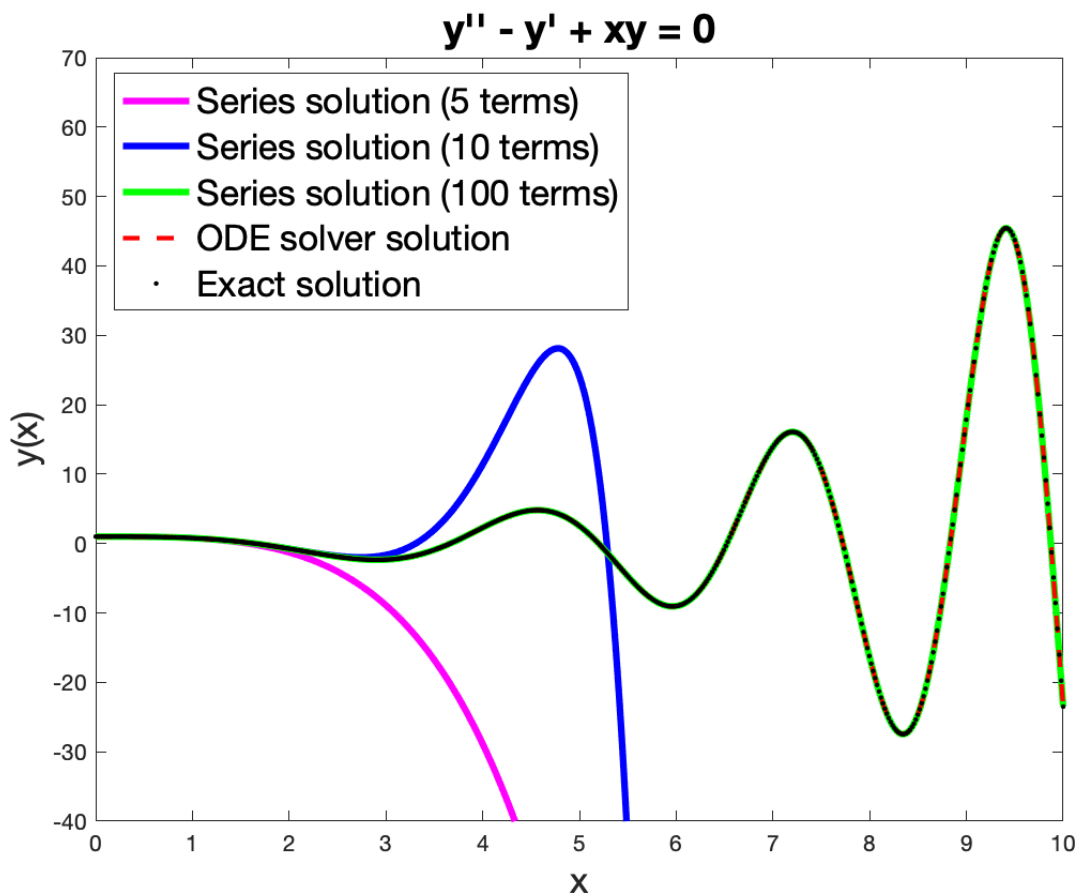


Figure 2: Plot of the solution to Problem 2.

2. Find a power series solution in power of x for the following ODE.

$$y'' - y' + xy = 0$$

- Find the first 10 terms (i.e. terms including a_{10}) in the series as expressions involving a_0 and a_1 . Show your work.
- Use the boundary conditions $y(0) = 1$ and $y'(0) = 0$ to get numeric values for a_0 and a_1 . Compute the remaining coefficients a_i numerically (using the recursion relation) and compare the numeric results with results of evaluating your analytic expression numerically. Your analytic coefficients should agree with the computed values almost exactly. Report both your analytic expressions and the numerically computed coefficients.
- Plot the solution over the interval $[0, 10]$ for the 5 term series, 10 term series and 100 term series. See Figure 2 for an example of what your figure should look like.

Note: Use the MATLAB code on the course webpage to get started with the numerical portion of this project.

3. (**Math 567 students**) Solve the ODE above using a numerical ODE solver (e.g. `ode45` in MATLAB) and code the exact solution, found, for example, on WolframAlpha. Plot both the exact solution

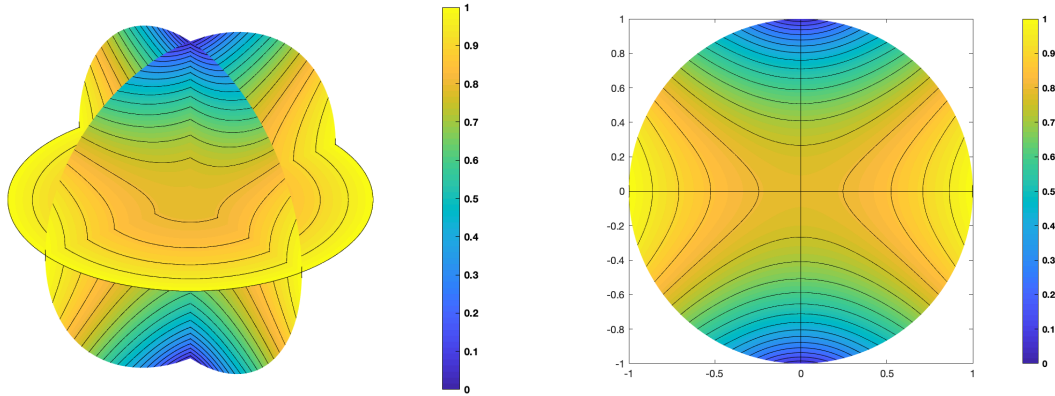


Figure 3: Solution to temperature distribution problem in the sphere (left) (Problem 5). Right plot shows the solution from the front.

(which involves the Airy function) and the numerically computed solution on the same plot as your series solution from Problem 2.

4. Rodrigues's Formula is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

Use this formula to derive Legendre polynomials $P_0(x)$ through $P_5(x)$ (in your book, equation 11', page 178).

5. We can use Legendre functions to compute the steady state temperature distribution of heat inside of a sphere. As in class, we will assume that the temperature on the sphere is symmetric with respect to longitude, but varies with latitude. We describe the sphere and its interior using coordinates (r, ϕ) where the north pole is at $\phi = 0$ and the south pole is at $\phi = \pi$ and r measures the distance from the center of the sphere. For the problem, we use the unit sphere, so that on the sphere, $r = 1$.

The solution to the temperature distribution problem in (r, ϕ) coordinates is given by

$$u(r, \phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos(\phi))$$

where $P_n(x)$ are the Legendre polynomials. The coefficients A_n are computed using the temperature distribution $u(1, \phi) = f(\phi)$ on the sphere. These coefficients are computed as

$$A_n = \frac{2n+1}{2} \int_0^\pi f(\phi) P_n(\cos(\phi)) \sin(\phi) d\phi$$

- If we hold the temperature on the sphere fixed to a constant temperature 1, what is the temperature distribution inside the sphere? To figure this out, compute the first few coefficients for $f(\phi) = 1$. Show your work.
- Let $f(\phi) = \sin(\phi)$. Using WolframAlpha, compute A_0 through A_{12} . Report the values that you get.
- Using the coefficients you computed in Problem 5, what is the temperature at the center of the sphere?

- (MATLAB) Plot the solution $u(1, \phi)$ for a four term series (using coefficients A_0 through A_4) and a 12 term series over the interval $[0, \pi]$. Compare your solution to $f(\phi) = \sin(\phi)$ and show that even for the four term series, your approximate solution matches the prescribed temperature $f(\phi)$ quite closely.
- (MATLAB, Math 527) Using the MATLAB code provided on the course website, compute the solution to the temperature distribution in the sphere. Turn in your plot of the solution, and code fragments showing where you modified the code by supplying various "TODO" items. You can see the final solution in Figure 3.