Homework # 3

Math 427/527

Note : Math 427 students may do the Math 527 questions for extra credit. You may work in pairs on this assignment, but pairs can only be two 427 students or two 527 students but not mixed pairs.

All plots must have axes labels, and a title. Also, be sure to use appropriate axis limits for each plot. Make your plots interesting!

1. Calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C.

   (a) $\mathbf{F} = [y^2, -x^2]$, $C : y = 4x^2$, from $(0, 0)$ to $(1, 4)$.

   (b) $\mathbf{F} = [x - y, y - z, z - x]$, $C : \mathbf{r} = [2\cos(t), t, 2\sin(t)]$, from $(2, 0, 0)$ to $(2, 2\pi, 0)$.

2. The area and circumference of an ellipse can be computed by evaluating line integrals of the form

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{dr}$$

where $C = \mathbf{r}(t)$ is the parameterization of the ellipse in the plane given by

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (a\cos(t), b\sin(t), 0), \qquad 0 \leq t \leq 2\pi, \qquad a > b$$

and

$$\mathbf{dr} = \mathbf{r}'(t)\ dt.$$

The unit tangent vector $\mathbf{t}$ and the unit normal vector $\mathbf{n}$ to the ellipse are given by

$$\mathbf{t} = \frac{\mathbf{r}'(t)}{g}$$
$$\mathbf{n} = \frac{g}{ab}\left(\mathbf{r}''(t) - (\mathbf{t} \cdot \mathbf{r}''(t))\mathbf{t}\right)$$

where $g = \| \mathbf{r}'(t) \|$.

   (a) Derive a formula for the circumference of the ellipse using

$$\mathbf{F}(\mathbf{r}) = \mathbf{t}.$$

   How is the formula you get related to the *complete elliptic integral of the second kind*? Use MATLAB's `ellipke` function to compute the circumference of the ellipse for $a = 5.4$ and $b = 2.3$. You should get the value 25.1808411069701386.

   (b) Derive a formula for the area of the ellipse using

$$\mathbf{F}(\mathbf{r}) = (\mathbf{H} \cdot \mathbf{n})\mathbf{t}$$

   for $\mathbf{H} = [x, 0, 0]$. Check that your formula is correct by using it to compute the area of a circle of radius R.

For both problems above, show all of your work.

3. Use Green's Theorem to compute the integral

$$\oint_C (y - \sin(x))dx + \cos(x)dy$$

over the curve described by the triangle with vertices $(0,0)$, $(\pi/2, 0)$ and $(\pi/2, 1)$.

4. Use Green's Theorem to find the value of

$$\oint \frac{\partial w}{\partial n} ds$$

for $w(x,y) = e^x \cos(x) + xy^3$ over the region $R : 1 \leq y \leq 10 - x^2,\ x \geq 0$.

5. (Surface integral) Evaluate

$$\int_S \mathbf{F} \cdot \mathbf{n}\, dA$$

where $\mathbf{F} = (-x^2, y^2, 0)$ and the surface S is given by the parameterization $\mathbf{r} = [u, v, 3u - 2v]$, $0 \leq u \leq 1.5$, $-2 \leq v \leq 2$.

6. (Surface integral) Evaluate

$$\iint_S (\cos(x) + \sin(x))\, dA$$

over the portion of $x + y + z = 1$ in the first octant.

7. (Math 527) Evaluate the integral

$$\int_S \mathbf{F} \cdot \mathbf{n}\, dA$$

for $\mathbf{F} = [0, \sin(y), \cos(z)]$ and S is the cylinder $x = y^2$, where $0 \leq y \leq \pi/4$ and $0 \leq z \leq y$.