Homework #1

Math 465/565

For problems that require programming, I suggest you use Matlab, Python, C, or Julia. If you'd like to use something other than one these languages, please ask me first.

You will be asked to turn in your homework electronically. I suggest one of the following modes :

- Use Jupyter Notebooks. You can include your code directly in the notebook, and compile or run from the notebook.
- Use Latex to turn in your results.
- Use Matlab Publish.

In any case, you will be required to turn in your code along with any discussion of the results.

Students in 465 can receive extra credit for doing problems designated as 565 only.

1. Let a be non-negative real number. Then, for any positive x_0, the sequence of iterates generated by the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad n = 0, 1, 2, \ldots \tag{1}$$

converges to $\sqrt{a}$.

Use this rule to write a routine to approximate $\sqrt{5}$. Use as your starting condition $x_0 = 5$, $k_{max} = 20$ and tolerance $\varepsilon = 1 \times 10^{-12}$.

2. We often need to be able to approximate the definite integral

$$\int_a^b f(x)\, dx \tag{2}$$

for some function $f(x)$ over a given interval $[a, b]$. One approximation is the Trapezoidal rule, given by

$$T_n(f) = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] \tag{3}$$

where $x_i = a + ih$, for $h = (b - a)/n$.

Write an algorithm that takes as input a function $f(x)$, and an interval $[a, b]$ and returns an approximation to the integral of $f(x)$.

Use your algorithm to find the value of the integral

$$\int_0^1 \frac{1}{1 + x^2} dx. \tag{4}$$

What are the values of your approximation for $n = 10$ and $n = 20$?

3. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$ be an n^{th} degree polynomial.

   (a) How many additions and multiplications are required to evaluate $P(x)$ for a given value of x? Treat each power as a repeated multiplication.

   (b) Can you devise an algorithm that reduces the required number of arithmetic operations? How many multiplications and how many additions are required by your algorithm?

   (c) (**Math 565 only**). Write a computer code that evaluates a polynomial, given a set of coefficients. Test your algorithm on the 5^{th} degree polynomial $p(x) = 3.1x^5 + \pi x^4 - x^3 + 4.7x + 4$. Your algorithm should require a minimal number of operations. Create a plot of the polynomial over the interval $[-2, 2]$. Explain why it is important to have an efficient method for polynomial evaluation.